**MATH168 - DIFFERENTIAL EQUATIONS I**

**Solutions of Differential Equations**

**Lecturer:** Dr. Peter Amoako-Yirenkyi

**Recommended Textbook:** Elementary differential Equation

The laws of the universe are written largely in the language of mathematics. Algebra is sufficient to solve many static problems, but the most interesting naturally phenomena involve change and are best described by equations that relate changing quantities. Many important and significant problems in engineering, the physical sciences, and the social sciences such as economics and business when formulated in mathematical terms require the determination of a function satisfying an equation containing the derivatives of unknown function. Such equations are called differential equation.

**General Solution**

**Definition 1** A formula that gives all solutions of a differential equation is called the *general solution* of the equation. A general solution to an *nth* order DE generally involves *n* independent arbitrary constants, each admitting a range of real values.

The number of distinct solutions depend on the order of the differential equation, for which they are equal. For example the, the differential equation \( \frac{dy}{dx} = -4 \) describes a straight line with a constant gradient of -4. There are however infinitely constant many straight lines that take the equation \( y = -4x + c \), where \( c \) is an arbitrary constant.

Since \( c \in \mathbb{R} \), \( y \) is a constantly many hence the differential equation \( \frac{dy}{dx} = -4 \) has many solutions, because \( c \) can assume any real number and the differential equation would be satisfied, hence the name *arbitrary constant*. The general solution is known as *family of solutions*.

**Determining a Differential Equation from a General Solution: The Elimination of Arbitrary Constant**

To determine a differential equation from a general solution, we need to eliminate the arbitrary constant. Methods for the elimination of arbitrary constants vary with the way in which the constants enter the relation. A method which is efficient for one problem may be poor for another. One fact persists throughout. Since each differentiation yields a new relation, number of derivatives that need to be used is the same as the number of arbitrary constants to be eliminated. We shall in each case determine the differential equation that is of order equal to the number of arbitrary constants in the given relation, consistent with that relation and free from arbitrary constants.
Example 1 Find the differential equation whose general solution is
\[ y = c_1 e^{-2x} + c_2 e^{3x} \]

Eliminating the arbitrary constants \(c_1\) and \(c_2\) from the relation:
\[ y = c_1 e^{-2x} + c_2 e^{3x} \]  \hspace{1cm} (3)

Since two constants are to be eliminated, obtain the two derivatives:
\[ y' = -2c_1 e^{-2x} + 3c_2 e^{3x} \]  \hspace{1cm} (4)
\[ y'' = 4c_1 e^{-2x} + 9c_2 e^{3x} \]  \hspace{1cm} (5)

The elimination of \(c_1\) from (3) and (4) yields: \(y'' + 2y' = 15c_2 e^{3x}\)
Hence \(y'' + 2y' = 3(y' + 2y)\) or \(y'' - y' - 6y = 0\)

Example 2 Find the differential equation whose general solution is \(y = c \sin x\), where \(c\) is an arbitrary constant.

Solution:
\[ y = c \sin x \] \hspace{1cm} (6)
\[ y' = c \cos x \] \hspace{1cm} (7)

There is one arbitrary constant, we then find the first derivative:
\[ \frac{dy}{dx} = c \cos x \] \hspace{1cm} (8)

From (6) and (8):
\[ c = \frac{y}{\sin x} = \frac{y'}{\cos x} \Rightarrow y' \sin x = y \cos x \Rightarrow y' \sin x - y \cos x = 0 \]

Example 3 Determine the differential equation whose general solution is \(y = c_1 e^x + c_2 e^{-x} + 2x\). See for extended version.

Solution:
There are two arbitrary constants, so we get equation of order 2. Hence we differentiate twice.

Let \(y = c_1 e^x + c_2 e^{-x} + 2x\) \hspace{1cm} (9)
\[ \Rightarrow \frac{dy}{dx} = c_1 e^x - c_2 e^{-x} + 2 \] \hspace{1cm} (10)
\[ \Rightarrow \frac{d^2 y}{dx^2} = c_1 e^x + c_2 e^{-x} \] \hspace{1cm} (11)

Add (1.8) and (1.9):
\[ \Rightarrow \frac{dy}{dx} + y = 2c_1 e^x + 2x + 2 \] \hspace{1cm} (12)

Add (1.9) and (1.10):
\[ \Rightarrow \frac{d^2 y}{dx^2} + \frac{dy}{dx} = 2c_1 e^x + 2 \] \hspace{1cm} (13)

The two equations above have the same term on the RHS, hence we equate the LHS, and we get:
\[ \frac{d^2 y}{dx^2} + \frac{dy}{dx} = \frac{dy}{dx} + y - 2x \]
This simplifies as:
\[ \frac{d^2 y}{dx^2} - y + 2x = 0 \]

\(^3\) Alternatively, (another) method for obtaining the differential equation in this example proceeds as follows. We know from a theorem in algebra (MATH152) that three equations (3), (4) and (5) considered as equations in the two unknowns \(c_1\) and \(c_2\) can have solutions only if:
\[
\begin{vmatrix}
-3 & e^{-2x} & e^{3x} \\
-9 & -2e^{-2x} & 3e^{3x} \\
-12 & 4e^{-2x} & 9e^{3x}
\end{vmatrix} = 0
\]  \hspace{1cm} (1)

Since \(e^{-2x}\) and \(e^{3x}\) cannot be zero for any \(x \in \mathbb{R}\), equation (1) may be rewritten, with the factors \(e^{-2x}\) and \(e^{3x}\) removed, as:
\[
\begin{vmatrix}
y & 1 & 1 \\
y' & -2 & 3 \\
y'' & 4 & 9
\end{vmatrix} = 0
\]  \hspace{1cm} (2)

From which the differential equation:
\(y'' - y' - 6y = 0\) follows immediately.

\(^4\) Alternatively, from (6) let \(c = \frac{y}{\sin x}\)

Differentiating w.r.t. \(x\): \(0 = y' \sin x - y \cos x\)
\[ \Rightarrow y' \sin x - y \cos x = 0 \]

\(^5\) Example 4 Find the differential equation whose general solution is
\[ y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 \]

Solution:
The equation has four arbitrary constants \((c_0, c_1, c_2, c_3)\); hence we need to differentiate four times to get a differential equation of the fourth order.

Let \(y = c_0 + c_1 x + c_2 x^2 + c_3 x^3\)
\[ \Rightarrow \frac{dy}{dx} = c_1 + 2c_2 x + 3c_3 x^2 \Rightarrow \frac{d^2 y}{dx^2} = 2c_2 + 6c_3 x \Rightarrow \frac{d^3 y}{dx^3} = 6c_3 \Rightarrow \frac{d^4 y}{dx^4} = 0, \text{which is the differential equation that we need.} \]
Example 5 Eliminate the constant $c$ from the equation:

$$(x - c)^2 + y^2 = c^2$$

Solution:

Direct differentiation of the relation yields: $2(x - c) + 2yy' = 0$ and simplified as $c = x + yy'$. Therefore using the original equation, we find that: $(yy')^2 + y^2 = (x + yy')^2$, or $y^2 = x^2 + 2xyy'$, which may be written in the form: $(x^2 - y^2)dx + 2xydy = 0$

Alternatively:

The equation $(x - c)^2 + y^2 = c^2$ may be put in the form $x^2 + y^2 - 2cx = 0$ or $\frac{x^2 + y^2}{x} = 2c$ then differentiation of both sides leads to:

$$\frac{x(2xdx + 2ydy) - (x^2 + y^2)dx}{x^2} = 0 \text{ or } (x^2 - y^2)dx + 2xydy = 0$$

Example 7 Eliminate $c$ from the equation $cxy + c^2x + 4 = 0$

Solution:

At once we get: $c(y + xy') + c^2 = 0$ Since $c \neq 0$, $c = -(y + xy')$ and substitution into the original equation leads us to the result:

$$x^3(y')^2 + x^2yy' + 4 = 0$$

Remark:

The general solution of an $n^{th}$ order ordinary differential equation could be expected to have $n$ arbitrary constants.

**Initial-value and Boundary-value Problems**

**Definition 2** (Subsidiary Conditions) Subsidiary Condition(s) is/are condition(s), or set of conditions, on the differential equation that will allow us to determine which solution that we are after.

**Definition 3** (Initial-Value Problem) Initial-Value Problem is the differential equation along with subsidiary conditions on the unknown function and its derivatives, all given at the same value of the independent variable. The subsidiary conditions are initial conditions.

**Definition 4** (Boundary-Value Problem) Boundary-Value Problem is the differential equation along with subsidiary conditions on the unknown function and its derivatives, which are given at more than one value of the independent variable. The subsidiary conditions are boundary conditions.

**Definition 5** An initial value problem (IVP) for an $n$th order DE includes a specification of the solution’s value and $n - 1$ derivatives at some point or a set of $n$ algebraic conditions at a common point:

$$y(x_0) = y_0, \quad \frac{dy}{dx}(x_0) = y_1, \ldots, \quad \frac{d^{n-1}y}{dx^{n-1}}(x_0) = y_{n-1}$$

---

\[\text{Example 6} \text{ Eliminate } B \text{ and } \alpha \text{ from the relation } x = B \cos(\omega t + \alpha) \text{ in which } \omega \text{ is a parameter (not to be eliminated)}\]

Solution:

First we obtain two derivatives of $x$ with respect to $t$:

$$\frac{dx}{dt} = -\omega B \sin(\omega t + \alpha) \quad \text{and} \quad \frac{d^2x}{dt^2} = -\omega^2 B \cos(\omega t + \alpha)$$

$$\Rightarrow \frac{d^2x}{dt^2} = -\omega^2 x, \text{ since } x = B \cos(\omega t + \alpha)$$

$$\Rightarrow \frac{d^2x}{dt^2} + \omega^2 x = 0$$

---

\[\text{Remark:} \text{ The problem } y'' + 2y' = e^t; y(0) = 1, \quad y'(1) = 1 \text{ is a boundary-value problem, because the two subsidiary conditions are given at the different values } x=0 \text{ and } x=1.\]

\[\text{Remark:} \text{ The problem } y'' + 2y' = e^t; y(\pi) = 1, \quad y'(\pi) = 2 \text{ is an initial-value problem, because the two subsidiary conditions are both given at } x = \pi.\]
Generally in applications, an IVP\(^9\) has a **unique solution** on some interval containing the initial value point. If an **initial condition** is given along with the differential equation, that is, a constraint of the form \(y = y_0\) when \(x = x_0\), then this information can be used to determine the particular value of \(c\). In this way, one particular solution (actual solution) can be selected from the family of solutions - the one that satisfies both the differential equation and the initial conditions.

**Example 8** \(y = c_1 e^{-x} + c_2 e^{3x}\) is a general solution of the differential equation \(y'' - 2y' - 3y = 0\).

Determine the particular solution of the initial conditions \(y = 3\) when \(x = 0\) and \(y' = 4\) when \(x = 0\).

**Solution:**
Since the differential equation is second order, we have a specification of the solution value: \(y(0) = 3\) and one derivative at a point \(y'(0) = 4\), that is 2 algebraic conditions at a common point: \(x = 0\)

\[
\begin{align*}
\text{let } y &= c_1 e^{-x} + c_2 e^{3x} \\
&\text{when } x = 0, y = 3
\end{align*}
\]

\(\Rightarrow 3 = c_1 e^{(0)} + c_2 e^{3(0)} = c_1 + c_2 \quad (16)\)

\[
\begin{align*}
y' &= -c_1 e^{-x} + 3c_2 e^{3x} \text{when } x = 0, y' = 4 \\
&\Rightarrow 4 = -c_1 e^{-(0)} + 3c_2^{(0)}
\end{align*}
\]

\(\Rightarrow 4 = -c_1 + 3c_2 \quad (17)\)

Solving equation (16) and (17) we have, \(c_1 = \frac{5}{4}\) and \(c_2 = \frac{7}{4}\)
the particular solution is given by:

\[
y = \frac{5}{4} e^{-x} + \frac{7}{4} e^{3x}\text{or } y = \frac{1}{4} \left(5e^{-x} + 7e^{3x}\right)
\]

**Example 9** Find a solution to the boundary-value problem \(^{10}\) \(y'' + 4y = 0; y(\pi/8) = 0, y(\pi/6) = 1\), if the general solution to the differential equation is \(y = c_1 \sin 2x + c_2 \cos 2x\)

**Solution:**
For \(x = \frac{\pi}{8}\), \(y = c_1 \sin 2\left(\frac{\pi}{8}\right) + c_2 \cos 2\left(\frac{\pi}{8}\right) = c_1 \sin \frac{\pi}{4} + c_2 \cos \frac{\pi}{4}\)

\(\Rightarrow y = c_1 \frac{\sqrt{2}}{2} + c_2 \frac{\sqrt{2}}{2}\)
To satisfy the condition \( y(\pi/8) = 0 \), we have:
\[
c_1 \frac{\sqrt{2}}{2} + c_2 \frac{\sqrt{2}}{2} = 0
\]
(20)

For \( x = \pi/6 \), \( y = c_1 \sin(\pi/6) + c_2 \cos(\pi/6) = c_1 \frac{\sqrt{3}}{2} + c_2 \frac{1}{2} \)
\[
\Rightarrow y = c_1 \frac{\sqrt{3}}{2} + c_2 \frac{1}{2}
\]

To satisfy the condition \( y(\pi/6) = 1 \), we have:
\[
c_1 \frac{\sqrt{3}}{2} + c_2 \frac{1}{2} = 1
\]
(21)

Solving (1.17) and (1.18) simultaneously, we get:
\[
c_1 = \frac{2}{\sqrt{3} - 1} \quad \text{and} \quad c_2 = -\frac{2}{\sqrt{3} - 1}
\]

Hence a particular solution is obtained:
\[
y = \frac{2}{\sqrt{3} - 1} (\sin 2x - \cos 2x)
\]
as the solution of the boundary-value problem.

**Example 11**  Show that the integral curves of the differential equation:
\((y - x^3)dx + (y^3 + x)dy = 0\) are given by the family \(y^4 + 4xy - x^4 = c\).

**Solution:** Apply implicit differentiation to the proposed family of integral curves to find \(y\);
\[
y^4 + 4xy - x^4 = c \Rightarrow 4y^3y' + (4xy' + 4y) - 4x^3 = 0
\]
\[
\Rightarrow 4y^3y' + (4xy' + 4y) - 4x^3 = 0 \Rightarrow y'(y^3 + x) + (y - x^3) = 0
\]
\[
\Rightarrow \frac{dy}{dx}(y^3 + x) + (y - x^3) = 0 \quad \frac{dy}{dx} = -\frac{(y - x^3)}{(y^3 + x)}
\]

This last equation\(^{11}\) can then be written in terms of the differentials \(dx\) and \(dy\)
\[
(y^3 + x)dy = -(y - x^3)dx \Rightarrow (y - x^3)dx + (y^3 + x)dy = 0
\]
(23)

**References**